AP CALCULUS AB and BC

Final Notes

Differentiation Formulas

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(fg) = fg' + gf'$ *Product rule*
3. $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$ *Quotient rule*
4. $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$ *Chain rule*
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\cos x) = -\sin x$
7. $\frac{d}{dx}(\tan x) = \sec^2 x$
8. $\frac{d}{dx}(\cot x) = -\csc^2 x$
9. $\frac{d}{dx}(\sec x) = \sec x \tan x$
10. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = a^x \ln a$
13. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
14. $\frac{d}{dx}(Arc\sin x) = \frac{1}{\sqrt{1-x^2}}$
15. $\frac{d}{dx}(Arc\tan x) = \frac{1}{1+x^2}$
16. $\frac{d}{dx}(Arc\sec x) = \frac{1}{|x|\sqrt{x^2-1}}$
17. $\frac{d}{dx}[c] = 0$
18. $\frac{d}{dx}[cf(x)] = cf'(x)$

Integration Formulas

1. $\int a\,dx = ax + C$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
3. $\int \frac{1}{x}dx = \ln|x| + C \quad \int \frac{du}{u} = \ln|u| + C \quad \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C \quad \int \frac{k}{x}dx = k\ln|x| + C$
4. $\int e^x\,dx = e^x + C \quad \int f'(x)e^{f(x)}dx = e^{f(x)} + C \quad \int e^{kx}dx = \frac{1}{k}e^{kx} + C$ {Shortcut}
5. $\int a^x dx = \frac{a^x}{\ln a} + C \quad \int a^u du = \frac{a^u}{\ln a} + C \quad \int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + C$
6. $\int \ln x\,dx = x\ln x - x + C$
7. $\int \sin x\,dx = -\cos x + C$
8. $\int \cos x\,dx = \sin x + C$
9. $\int \tan x\,dx = \ln|\sec x| + C$ or $-\ln|\cos x| + C$
10. $\int \cot x\,dx = \ln|\sin x| + C$

11. $\int \sec x\, dx = \ln\left|\sec x + \tan x\right| + C$

12. $\int \csc x\, dx = -\ln\left|\csc x + \cot x\right| + C$

13. $\int \sec^2 x\, dx = \tan x + C$

14. $\int \sec x \tan x\, dx = \sec x + C$

15. $\int \csc^2 x\, dx = -\cot x + C$

16. $\int \csc x \cot x\, dx = -\csc x + C$

17. $\int \tan^2 x\, dx = \tan x - x + C$

18. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} Arc\tan\left(\frac{x}{a}\right) + C$

19. $\int \frac{dx}{\sqrt{a^2 - x^2}} = Arc\sin\left(\frac{x}{a}\right) + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} Arc\sec\frac{|x|}{a} + C = \frac{1}{a} Arc\cos\left|\frac{a}{x}\right| + C$

Shortcuts

$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$

$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C$

$\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + C$

$\int \sec(kx)\tan(kx)\,dx = \frac{1}{k}\sec(kx) + C$

$\int \csc^2(kx)\,dx = -\frac{1}{k}\cot(kx) + C$

$\int \csc(kx)\cot(kx)\,dx = -\frac{1}{k}\csc(kx) + C$

$\int \tan(kx)\,dx = -\frac{1}{k}\ln\left|\cos(kx)\right| + C = \frac{1}{k}\ln\left|\sec(kx)\right| + C$

$\int \sec(kx)\,dx = \frac{1}{k}\ln\left|\sec(kx) + \tan(kx)\right| + C$

$\int \csc(kx)\,dx = -\frac{1}{k}\ln\left|\csc(kx) + \cot(kx)\right| + C$

$\int \cot(kx)\,dx = \frac{1}{k}\ln\left|\sin(kx)\right| + C$

Formulas, Concepts, and Theorems

1. **Limits and Continuity:**

A function $y = f(x)$ is continuous at $x = a$ if

i). $f(a)$ exists

ii). $\lim_{x\to a} f(x)$ exists

iii). $\lim_{x\to a} f(x) = f(a)$

Otherwise, f is discontinuous at $x = a$.

The limit $\lim_{x\to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal – that is,

$$\lim_{x\to a} f(x) = L \to \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$$

2. **Even and Odd Functions**

1. A function $y = f(x)$ is even if $f(-x) = f(x)$ for every x in the function's domain. Every even function is symmetric about the y-axis.
2. A function $y = f(x)$ is odd if $f(-x) = -f(x)$ for every x in the function's domain. Every odd function is symmetric about the origin.

3. **Periodicity**

A function $f(x)$ is periodic with period $p\ (p > 0)$ if $f(x+p) = f(x)$ for every value of x .

Note: The period of the function $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ is $\frac{2\pi}{|B|}$.

The amplitude is $|A|$. The period of $y = \tan x$ is π.

4. **Intermediate-Value Theorem**

A function $y = f(x)$ that is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$.

Note: If f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a,b).

5. **Limits of Rational Functions as $x \to \pm\infty$**

i). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{x^2 - 2x}{x^3 + 3} = 0$

ii). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is infinite if the degrees of $f(x) >$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{x^3 + 2x}{x^2 - 8} = \infty$

iii). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{2x^2 - 3x + 2}{10x - 5x^2} = -\frac{2}{5}$

6. Horizontal and Vertical Asymptotes

1. A line $y = b$ is a **horizontal asymptote** of the graph $y = f(x)$ if either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.(Compare degrees of functions in fraction)
2. A line $x = a$ is a **vertical asymptote** of the graph $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$ (Values that make the denominator 0 but not numerator)

7. Average and Instantaneous Rate of Change

i). ***Average Rate of Change***: If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is $\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$.{old fashioned way of finding slope}

ii). ***Instantaneous Rate of Change***: If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$. {new fashioned way of finding slope}

8. Definition of Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The latter definition of the derivative is the instantaneous rate of change of $f(x)$ with respect to x at $x = a$. {i.e. the slope at $x = a$ }

Geometrically, ***the derivative of a function at a point is the slope of the tangent line to the graph of the function at that point.***

9. The Number e as a limit ***BC only

i). $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

ii). $\lim_{n \to 0} (1 + n)^{1/n} = e$

10. Rolle's Theorem (this is a weak version of the MVT)

If f is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a, b) such that $f'(c) = 0$.

11. **Mean Value Theorem**

If f is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one number c in (a,b) such that $\frac{f(b)-f(a)}{b-a}=f'(c)$.

12. **Extreme-Value Theorem**

If f is continuous on a closed interval $[a,b]$, then $f(x)$ has both a maximum and minimum on $[a,b]$.

13. ***Absolute Mins and Maxs***: To find the maximum and minimum values of a function $y=f(x)$ on a CLOSED interval, locate

1. the points where $f'(x)$ is zero or where $f'(x)$ fails to exist. {Must be within the given interval}
2. the end points, if any, on the domain of $f(x)$.
3. Plug those values into $f(x)$ to see which gives you the max and which gives you this min values (the x-value is where that value occurs)

Note: These are the only candidates for the value of x where $f(x)$ may have a maximum or a minimum.

14. **Increasing and Decreasing**:

1. If $f'(x)>0$ for every x in (a,b), then f is increasing on $[a,b]$.
2. If $f'(x)<0$ for every x in (a,b), then f is decreasing on $[a,b]$.

Relative mins and maxs:

1. Relative Max – if f' switches from positive to negative at $x=a$, then $(a, f(a))$ is a relative max of f.
2. Relative Min – if f' switches from negative to positive at $x=a$, then $(a, f(a))$ is a relative min of f.

15. **Concavity:**

1. If $f''(x)>0$ in (a,b), $\{f' \text{ is increasing}\}$ then f is concave upward in (a,b).
2. If $f''(x)<0$ in (a,b), $\{f' \text{ is decreasing}\}$ then f is concave downward in (a,b).

To locate the ***points of inflection*** of $y=f(x)$, find the points where $f''(x)=0$ or where $f''(x)$ fails to exist $\{\text{or the } x-\textit{values} \text{ where } f' \text{ has mins and maxs}\}$. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x)<0$ on one side and $f''(x)>0$ on the other.

16a. **Differentiability and Continuity Theorem:** If a function is differentiable at point $x=a$, it is continuous at that point. The converse is false, in other words, continuity does not imply differentiability.

16b. **Local Linearity and Linear Approximations {aka "Tangent lines"}**

The linear approximation to $f(x)$ near $x=x_0$ is given by $y=f(x_0)+f'(x_0)(x-x_0)$ for x sufficiently close to x_0. In other words, find the equation of the tangent line at $(x_0, f(x_0))$ and use that equation to approximate the value at the value you need an estimate for.

17. ***Dominance and Comparison of Rates of Change (BC topic only)**

Logarithm functions grow slower than any power function (x^n).

Among power functions, those with higher powers grow faster than those with lower powers.
All power functions grow slower than any exponential function $(a^x, a>1)$.

Among exponential functions, those with larger bases grow faster than those with smaller bases.
We say, that as $x\to\infty$:

1. $f(x)$ grows faster than $g(x)$ if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty$ or if $\lim_{x\to\infty}\frac{g(x)}{f(x)}=0$.

If $f(x)$ grows faster than $g(x)$ as $x\to\infty$, then $g(x)$ grows slower than $f(x)$ as $x\to\infty$.

2. $f(x)$ and $g(x)$ grow at the same rate as $x\to\infty$ if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=L\neq 0$ (L is finite and nonzero).

For example,

1. e^x grows faster than x^3 as $x\to\infty$ since $\lim_{x\to\infty}\frac{e^x}{x^3}=\infty$

2. x^4 grows faster than $\ln x$ as $x\to\infty$ since $\lim_{x\to\infty}\frac{x^4}{\ln x}=\infty$

3. x^2+2x grows at the same rate as x^2 as $x\to\infty$ since $\lim_{x\to\infty}\frac{x^2+2x}{x^2}=1$

To find some of these limits as $x\to\infty$, you may use the graphing calculator. Make sure that an appropriate viewing window is used.

18. ***L'Hôpital's Rule***

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

19. **Inverse function**

1. If f and g are two functions such that $f(g(x))=x$ for every x in the domain of g and $g(f(x))=x$ for every x in the domain of f, then f and g are inverse functions of each other.
2. A function f has an inverse if and only if no horizontal line intersects its graph more than once.
3. If f is strictly either increasing or decreasing in an interval, then f has an inverse.
4. If f is differentiable at every point on an interval I, and $f'(x)\neq 0$ on I, then $g=f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(I)$ and if the point (a,b) is on $f(x)$, then the point (b,a) is on $g=f^{-1}(x)$; furthermore $g'(b)=\frac{1}{f'(a)}$.

	$f(x)$	$f^{-1}(x)$
Point	(a,b)	(b,a)
Slope	$f'(a)=k$	$(f^{-1})'(b)=\frac{1}{k}$

20. **Properties of** $y=e^x$

1. The exponential function $y=e^x$ is the inverse function of $y=\ln x$.
2. The domain is the set of all real numbers, $-\infty<x<\infty$.
3. The range is the set of all positive numbers, $y>0$.
4. $\frac{d}{dx}(e^x)=e^x$ and $\frac{d}{dx}\left(e^{f(x)}\right)=f'(x)e^{f(x)}$
5. $e^{x_1}\cdot e^{x_2}=e^{x_1+x_2}$
6. $y=e^x$ is continuous, increasing, and concave up for all x.
7. $\lim_{x\to\infty}e^x=+\infty$ and $\lim_{x\to-\infty}e^x=0$.
8. $e^{\ln x}=x$, for $x>0$; $\ln(e^x)=x$ for all x.

21. **Properties of** $y = \ln x$

1. The domain of $y = \ln x$ is the set of all positive numbers, $x > 0$.
2. The range of $y = \ln x$ is the set of all real numbers, $-\infty < y < \infty$.
3. $y = \ln x$ is continuous and increasing everywhere on its domain.
4. $\ln(ab) = \ln a + \ln b$. {Very useful and necessary for derivatives.}
5. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$. {Very useful and necessary for derivatives.}
6. $\ln a^r = r \ln a$. {Very useful and necessary for derivatives.}
7. $y = \ln x < 0$ if $0 < x < 1$.
8. $\lim_{x\to\infty} \ln x = +\infty$ and $\lim_{x\to 0^+} \ln x = -\infty$.
9. $\log_a x = \frac{\ln x}{\ln a}$
10. $\frac{d}{dx}\left(\ln f(x)\right) = \frac{f'(x)}{f(x)}$ and $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$

22. **Trapezoidal Rule**

If a function f is continuous on the closed interval $[a,b]$ where $[a,b]$ has been ***equally*** partitioned into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, ...$[x_{n-1}, x_n]$, each length $\frac{b-a}{n}$, then

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)\right],$$ which is equivalent to $\frac{1}{2}(Leftsum + Rightsum)$

23a. **Definition of Definite Integral as the Limit of a Sum**

Suppose that a function $f(x)$ is continuous on the closed interval $[a,b]$. Divide the interval into n equal subintervals, of length $\Delta x = \frac{b-a}{n}$. Choose one number in each subinterval, in other words, x_1 in the first, x_2 in the second, …, x_k in the k th,…, and x_n in the n th. Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)dx = F(b) - F(a).$$

23b. **Properties of the Definite Integral**

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

i). $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c.

ii). $\int_a^a f(x)\,dx = 0$

iii). $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

iv). $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where f is continuous on an interval containing the numbers a, b, and c.

v). If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

vi). If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

vii). If $f(x) \geq 0$ on $[a,b]$, then $\int_{a}^{b} f(x)\,dx \geq 0$

viii). If $g(x) \geq f(x)$ on $[a,b]$, then $\int_{a}^{b} g(x)\,dx \geq \int_{a}^{b} f(x)\,dx$

24. **Fundamental Theorem of Calculus:**

$$\int_{a}^{b} f(x)\,dx = F(b) - F(a), \text{ where } F'(x) = f(x).$$

25. **Second Fundamental Theorem of Calculus (Steve's Theorem):**

$$\frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x) \quad \text{or} \quad \frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = g'(x)\,f(g(x)) - h'(x)f(h(x))$$

26. **Velocity, Speed, and Acceleration**

1. The velocity of an object tells how fast it is going and in which direction. Velocity is an instantaneous rate of change. If velocity is positive (graphically above the "x"-axis), then the object is moving away from its point of origin. If velocity is negative (graphically below the "x"-axis), then the object is moving back towards its point of origin. If velocity is 0 (graphically the point(s) where it hits the "x"-axis), then the object is not moving at that time.
2. The speed of an object is the absolute value of the velocity, $|v(t)|$. It tells how fast it is going disregarding its direction.
   The speed of a particle increases (speeds up) when the velocity and acceleration have the same signs. The speed decreases (slows down) when the velocity and acceleration have opposite signs.
3. The acceleration is the instantaneous rate of change of velocity – it is the derivative of the velocity – that is, $a(t) = v'(t)$. Negative acceleration (deceleration) means that the velocity is decreasing (i.e. the velocity graph would be going down at that time), and vice-versa for acceleration increasing. The acceleration gives the rate at which the velocity is changing.

Therefore, if x is the displacement of a moving object and t is time, then:

i) velocity = $v(t) = x'(t) = \frac{dx}{dt}$

ii) acceleration = $a(t) = x''(t) = v'(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

iii) $v(t) = \int a(t)\,dt$

iv) $x(t) = \int v(t)\,dt$

Note: The average velocity of a particle over the time interval from t_0 to another time t, is

Average Velocity = $\frac{\text{Change in position}}{\text{Length of time}} = \frac{s(t) - s(t_0)}{t - t_0}$, where $s(t)$ is the position of the particle at time t or $\frac{1}{b-a}\int_{a}^{b} v(t)\,dt$ if given the velocity function.

27. The **<u>average value</u>** of $f(x)$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$. (Note, this gives the Average of what is given – i.e. given velocity, this would find average velocity)

28. **<u>Area Between Curves</u>**

If f and g are continuous functions such that $f(x) \ge g(x)$ on $[a,b]$, then area between the curves is $\int_a^b [f(x)-g(x)]\,dx$ or $\int_a^b [top-bottom]\,dx$ or $\int_c^d [right-left]dy$.

29. *****<u>Integration By "Parts"</u>**

If $u = f(x)$ and $v = g(x)$ and if $f'(x)$ and $g'(x)$ are continuous, then

$\int u\,dv = uv - \int v\,du$.

<u>Note</u>: The goal of the procedure is to choose u and dv so that $\int v\,du$ is easier to solve than the original problem.

<u>Suggestion</u>:

When "choosing" u, remember **L.I.P.E.T**, where **L** is the logarithmic function, **I** is an inverse trigonometric function, **P** is a polynomial function, **E** is the exponential function, and **T** is a trigonometric function. Just choose u as the first expression in **L.I.P.E.T** (and dv will be the remaining part of the integrand). For example, when integrating $\int x\ln x\,dx$, choose $u = \ln x$ since **L** comes first in **L.I.P.E.T**, and $dv = x\,dx$. When integrating $\int xe^x\,dx$, choose $u = x$, since x is an algebraic function, and **A** comes before **E** in **L.I.P.E.T**, and $dv = e^x dx$. One more example, when integrating $\int x\,Arc\tan(x)\,dx$, let $u = Arc\tan(x)$, since **I** comes before **A** in **L.I.P.E.T**, and $dv = x\,dx$. {Don't forget tabular form for parts when u is a polynomial}

30. **<u>Volume of Solids of Revolution</u>** (rectangles drawn perpendicular to the axis of revolution)

- Revolving around a horizontal line (y=# or x-axis) where $a \le x \le b$:

  Axis of Revolution and the region being revolved:

  $$V = \pi\int_a^b \left(furthest\ from\ a.r. - a.r.\right)^2 - \left(closest\ to\ a.r. - a.r.\right)^2 dx$$

- Revolving around a vertical line (x=# or y-axis) where $c \le y \le d$ (or use Shell Method):

  Axis of Revolution and the region being revolved:

  $$V = \pi\int_c^d \left(furthest\ from\ a.r. - a.r.\right)^2 - \left(closest\ to\ a.r. - a.r.\right)^2 dy$$

30b. **<u>Volume of Solids with Known Cross Sections</u>**

1. For cross sections of area $A(x)$, taken perpendicular to the x-axis, volume = $\int_a^b A(x)\,dx$.

Cross-sections {if only one function is used then just use that function, if it is between two functions use *top-bottom if perpendicular to the x-axis or right-left if perpendicular to the y-axis*} mostly all the same only varying by a constant, with the only exception being the rectangular cross-sections:

- Square cross-sections:

  $$V = \int_a^b \left(top\ function - bottom\ function\right)^2 dx$$

- Equilateral cross-sections:

  $$V = \frac{\sqrt{3}}{4}\int_a^b \left(top\ function - bottom\ function\right)^2 dx$$

- Isosceles Right Triangle cross-sections (hypotenuse in the *xy* plane):

$$V=\frac{1}{4}\int_a^b\left(top\ function-bottom\ function\right)^2 dx$$

- Isosceles Right Triangle cross-sections (leg in the *xy* plane):

$$V=\frac{1}{2}\int_a^b\left(top\ function-bottom\ function\right)^2 dx$$

- Semi-circular cross-sections:

$$V=\frac{\pi}{8}\int_a^b\left(top\ function-bottom\ function\right)^2 dx$$

- Rectangular cross-sections (height function or value must be given or articulated somehow – notice no "square" on the {top – bottom} part):

$$V=\int_a^b\left(top\ function-bottom\ function\right)\bullet(height\ function\ \text{or}\ value)dx$$

- Circular cross-sections with the diameter in the *xy* plane:

$$V=\frac{\pi}{4}\int_a^b\left(top\ function-bottom\ function\right)^2 dx$$

- Square cross-sections with the diagonal in the *xy* plane:

$$V=\frac{1}{2}\int_a^b\left(top\ function-bottom\ function\right)^2 dx$$

2. For cross sections of area $A(y)$, taken perpendicular to the y-axis, volume = $\int_a^b A(y)\,dy$.

30c. *****Shell Method** (used if function is in terms of x and revolving around a vertical line) where $a\le x\le b$: ***<u>{This is a concept not used in either AB or BC but may be handy for college}</u>***

$$V=2\pi\int_a^b r(x)h(x)dx$$

$r(x)=x$ if a.r. is y-axis $(x=0)$

$r(x)=(x-a.r.)$ if a.r. is to the left of the region

$r(x)=(a.r.-x)$ if a.r. is to the right of the region

$h(x)=f(x)$ if only revolving with one function

$h(x)=\left(top-bottom\right)$ if revolving the region between two functions

31. **<u>Solving Differential Equations: Graphically and Numerically</u>**

<u>Slope Fields</u>

At every point $(x,\ y)$ a differential equation of the form $\frac{dy}{dx}=f(x,\ y)$ gives the slope of the member of the family of solutions that contains that point. A slope field is a graphical representation of this family of curves. At each point in the plane, a short segment is drawn whose slope is equal to the value of the derivative at that point. These segments are tangent to the solution's graph at the point.

Diff EQ Solutions:

1. Separate Variables
2. Integrate Both Sides (+C on the independent variable side)
3. If $\ln$ on the dependent variable side, solve for y first, then solve for C. If no $\ln$ on the dependent variable side, solve for C, then solve for y.

***Euler's Method (BC topic)

Euler's Method is a way of approximating points on the solution of a differential equation $\frac{dy}{dx} = f(x, y)$. The calculation uses the tangent line approximation to move from one point to the next. That is, starting with the given point (x_1, y_1) – the initial condition, the point $(x_1 + \Delta x, y_1 + f'(x_1, y_1)\Delta x)$ approximates a nearby point on the solution graph. This aproximation may then be used as the starting point to calculate a third point and so on. The accuracy of the method decreases with large values of Δx. The error increases as each successive point is used to find the next.

$(x, y): given$	$\frac{dy}{dx}: given$	$\Delta x: given$	$\Delta y = \frac{dy}{dx}\Delta x$	$(x+\Delta x, y+\Delta y)$
Start again				

32. ***<u>Logistics (BC topic)</u>**

1. Rate is jointly proportional to its size and the difference between a fixed positive number (L) and its size.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \text{ OR } \frac{dy}{dt} = ky(M - y) \text{ which yields}$$

$$y = \frac{M}{1 + Ce^{-Mkt}} \quad \text{through separation of variables}$$

2. $\lim_{t\to\infty} y = M$; M = carrying capacity (Maximum); horizontal asymptote
3. y-coordinate of inflection point is $\frac{L}{2}$, i.e. when it is growing the fastest (or max rate).

32(a). ***<u>Decomposition</u>**:

Steps:

1. Use Long Division first if the degree of the Numerator is equal or more than the Denominator to get $\int \frac{N(x)}{D(x)}dx = \int q(x)dx + \int \frac{r(x)}{D(x)}dx$
2. For the second integral, factor $D(x)$ completely into Linear factors to get $\frac{r(x)}{D(x)} = \frac{A}{linearfactor\#1} + \frac{B}{linearfactor\#2} + ...$
3. Multiply both sides by $D(x)$ to eliminate the fractions
4. Choose your x-values wisely so that you can easily solve for A, B, C, etc
5. Rewrite your integral that has been decomposed and integrate everything.

33. ***<u>Definition of Arc Length</u>**

If the function given by $y = f(x)$ represents a smooth curve on the interval $[a,b]$, then the arc length of f between a and b is given by $s = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$.

34. ***<u>Improper Integral</u>**

$\int_a^b f(x)\,dx$ is an improper integral if

1. f becomes infinite at one or more points of the interval of integration, or
2. one or both of the limits of integration is infinite, or
3. both (1) and (2) hold.

35. ***<u>**Parametric Form of the Derivative**</u>**

If a smooth curve **C** is given by the parametric equations $x = f(x)$ and $y = g(t)$, then the slope of the curve **C** at (x, y) is $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt},\ \frac{dx}{dt} \neq 0$.

<u>Note</u>: The second derivative, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right] \div \frac{dx}{dt}$.

36. ***<u>**Arc Length in Parametric Form**</u>**

If a smooth curve **C** is given by $x = f(t)$ and $y = g(t)$ and these functions have continuous first derivatives with respect to t for $a \le t \le b$, and if the point $P(x, y)$ traces the curve exactly once as t moves from $t = a$ to $t = b$, then the length of the curve is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\, dt.$$

$$\text{speed} = \sqrt{\left(\text{f}'(\text{t})\right)^2 + \left(\text{g}'(\text{t})\right)^2}$$

37. ***<u>**Vectors**</u>**

Velocity, speed, acceleration, and direction of motion in Vector form

- position vector is $r(t) = \langle x(t), y(t) \rangle$
- velocity vector is $v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$
- speed is the magnitude of velocity because $speed = |v(t)| = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
- acceleration vector is $a(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$
- the direction of motion is based on the velocity vector and the signs on its components

Displacement and distance travelled in vector form

- Displacement in vector form $\left\langle \int_a^b v_1(t)\,dt, \int_a^b v_2(t)\,dt \right\rangle$
- Final position in vector form $\left(x_1(a) + \int_a^b v_1(t)\,dt, x_2(a) + \int_a^b v_2(t)\,dt \right)$
- Distance travelled from

  $t = a$ to $t = b$ is given by $\int_a^b |v(t)|\,dt = \int_a^b \sqrt{\left(v_1(t)\right)^2 + \left(v_2(t)\right)^2}\,dt$

38. ***<u>**Polar Coordinates**</u>**

1. <u>**Cartesian vs. Polar Coordinates**</u>. The polar coordinates (r, θ) are related to the Cartesian coordinates (x, y) as follows:

   $x = r\cos\theta$ and $y = r\sin\theta$ $\qquad\qquad$ $\tan\theta = \frac{y}{x}$ and $x^2 + y^2 = r^2$

2. To find the points of intersection of two polar curves, and solve for θ. Check separately to see if the origin lies on both curves, i.e. if r can be 0. Sketch the curves.
3. <u>**Area in Polar Coordinates**</u>: If f is continuous and nonnegative on the interval $[\alpha, \beta]$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_\alpha^\beta \left(f(\theta)\right)^2 d\theta = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

4. **Derivative of Polar function**: Given $r = f(\theta)$, to find the derivative, use parametric equations.

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

Then $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

5. **Arc Length in Polar Form**: $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

6. **Area for a region inside r_1 and outside r_2** is $\frac{1}{2}\int_{\theta_1}^{\theta_2} r_1^2 - r_2^2 \, d\theta$

7. **Area shared by two polar curves r_1 and r_2** is given by $\frac{1}{2}\int_{\theta_1}^{\theta_2} r_1^2 d\theta + \frac{1}{2}\int_{\theta_2}^{\theta_3} r_2^2 d\theta$ {remember to use symmetry if able to}

8. **Increasing or decreasing:** calculate $\frac{dr}{d\theta}$ and determine if it is positive or negative.

39. ***__Sequences and Series__**

1. If a sequence $\{a_n\}$ has a limit L, that is, $\lim_{n\to\infty} a_n = L$, then the sequence is said to <u>converge</u> to L. If there is no limit, the series <u>diverges</u>. If the sequence $\{a_n\}$ converges, then its limit is unique. Keep in mind that

$$\lim_{n\to\infty}\frac{\ln n}{n} = 0; \qquad \lim_{n\to\infty} x^{(1/n)} = 1; \qquad \lim_{n\to\infty}\sqrt[n]{n} = 1; \qquad \lim_{n\to\infty}\frac{x^n}{n!} = 0.$$

These limits are useful and arise frequently.

2. The **<u>harmonic series</u>** $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges; the **<u>geometric series</u>** $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{\text{first term by using the initial } n \text{ value on the sigma}}{1-(\text{the "value" that has the power})} = \frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$ and $a \neq 0$.

3. The **<u>p-series</u>** $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

4. **<u>Limit Comparison Test</u>:** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a series of nonnegative terms, with $a_n \neq 0$ for all sufficiently large n, and suppose that $\lim_{n\to\infty}\frac{b_n}{a_n} = c > 0$. Then the two series either both converge or both diverge.

5. **Alternating Series:** Let $\sum_{n=1}^{\infty} a_n$ be a series such that

i) the series is alternating

ii) $\left| a_{n+1} \right| \le \left| a_n \right|$ for all n, and

iii) $\lim_{n\to\infty} a_n = 0$

Then the series *converges*.

Alternating Series Remainder (Error Bound): (This is used on an alternating series instead of the LaGrange Error Bound)The remainder R_N is less than (or equal to) the first neglected term {absolute value of the next term in the series that was not used}

$$|R_n| \le |a_{n+1}|$$

6. **The *n*-th Term Test for Divergence:** If $\lim_{n\to\infty} a_n \ne 0$, then the series diverges.

Note that the converse is *false*, that is, if $\lim_{n\to\infty} a_n = 0$, the series may or may not converge.

7. A series $\sum a_n$ is **absolutely convergent** if the series $\sum |a_n|$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ does not converge, then the series is conditionally convergent. Keep in mind that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

8. **Direct Comparison Test:** If $0 \le a_n \le b_n$ for all sufficiently large n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

9. **Integral Test:** If $f(x)$ is a positive, continuous, and decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ will converge if the improper integral $\int_1^{\infty} f(x)\,dx$ converges. If the improper integral $\int_1^{\infty} f(x)\,dx$ diverges, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

10. **Ratio Test:** Let $\sum a_n$ be a series with nonzero terms.

i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then the series converges absolutely.

ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then the series is divergent.

iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the test is inconclusive (and another test must be used).

11. **Power Series:** A power series is a series of the form

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$ or

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots + c_n (x-a)^n + \ldots$ in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ in the polynomial are constants. The set of all numbers x for which the power series converges is called the interval of convergence.

12. **Taylor Series**: Let f be a function with derivatives of all orders throughout some intervale containing a as an interior point. Then the Taylor series generated by f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots$$

The remaining terms after the term containing the nth derivative can be expressed as a remainder to Taylor's Theorem:

Taylor's Theorem

$$f(x)=f(a)+\sum_{1}^{n} f^{(n)}(a)(x-a)^n+R_n(x) \text{ where } R_n(x) \text{ is the remainder terms}$$

Lagrange's form of the remainder: $|f(x)-P_n(x)|=|R_n x|=\frac{M}{(n+1)!}|(x-a)|^{n+1}$,

where $a<c<x$ and M is found by finding $f^{(n+1)}$ and its max value on the interval of $[a,x]$

The series will converge for all values of x for which the remainder approaches zero as $x \to \infty$.

13. **Frequently Used Series and their Interval of Convergence**

$$\frac{1}{1-x}=1+x+x^2+\ldots+x^n+\ldots=\sum_{n=0}^{\infty} x^n, \ |x|<1$$

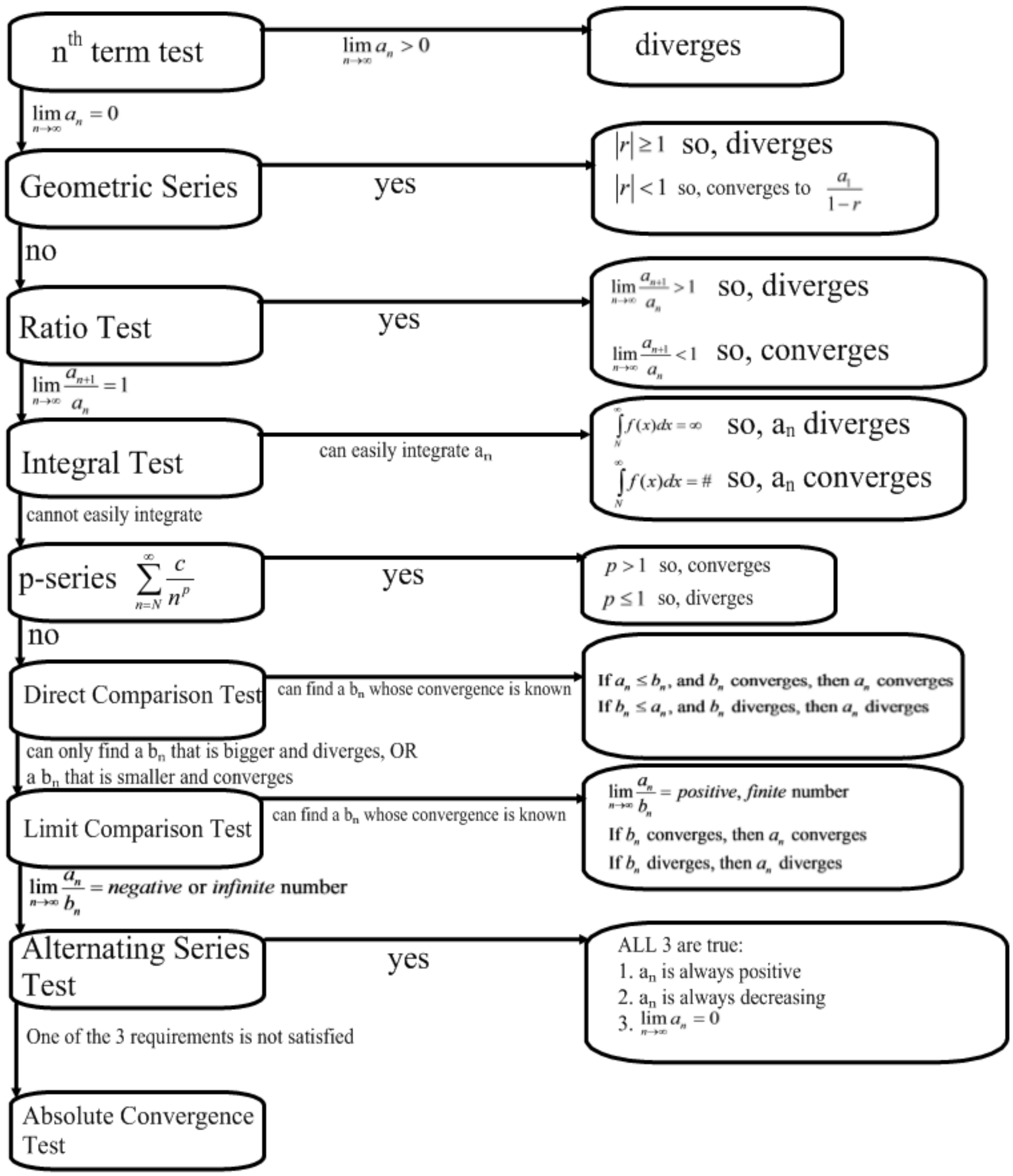
$$e^x=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!}, \ |x|<\infty$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}, \ |x|<\infty$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)x^{2n}}{(2n)!}, \ |x|<\infty$$

14. **Interval of Convergence (IOC) and Radius of Convergence (ROC).**

1. Apply the Ratio Test to $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}$. If the answer is:
    a. 0, then the ROC is infinite so it converges for all x
    b. ∞, then the ROC is 0, so it only converges at its center
    c. $|\text{expression}|$, then the ROC is found by setting $|\text{expression}|<1$ and getting it in the form $|x-a|<R$
    d. To find the IOC, get x by itself so that $-R+a<x<R+a$. Plug each endpoint in the original a_n for x, simplify, and use a convergence test to see if the series converges. If it does, add the "equal to" part to the inequality.



n^{th} term test
$\lim_{n\to\infty} a_n > 0$
diverges
$\lim_{n\to\infty} a_n = 0$
Geometric Series
yes
$|r| \geq 1$ so, diverges
$|r| < 1$ so, converges to $\frac{a_1}{1-r}$
no
Ratio Test
yes
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ so, diverges
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ so, converges
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$
Integral Test
can easily integrate a_n
$\int_N^\infty f(x)dx = \infty$ so, a_n diverges
$\int_N^\infty f(x)dx = \#$ so, a_n converges
cannot easily integrate
p-series $\sum_{n=N}^{\infty} \frac{c}{n^p}$
yes
$p > 1$ so, converges
$p \leq 1$ so, diverges
no
Direct Comparison Test
can find a b_n whose convergence is known
If $a_n \leq b_n$, and b_n converges, then a_n converges
If $b_n \leq a_n$, and b_n diverges, then a_n diverges
can only find a b_n that is bigger and diverges, OR
a b_n that is smaller and converges
Limit Comparison Test
can find a b_n whose convergence is known
$\lim_{n\to\infty} \frac{a_n}{b_n}$ = ***positive***, ***finite*** number
If b_n converges, then a_n converges
If b_n diverges, then a_n diverges
$\lim_{n\to\infty} \frac{a_n}{b_n}$ = ***negative*** or ***infinite*** number
Alternating Series Test
yes
ALL 3 are true:
1. a_n is always positive
2. a_n is always decreasing
3. $\lim_{n\to\infty} a_n = 0$
One of the 3 requirements is not satisfied
Absolute Convergence Test